Determination of the stochastic NPV and real options

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In this investigation a methodology sets out that standardizes the use of real options with jumps to the Net Present Value (VAN), besides to tie the cash flow of the company analyzed to a structure of term created from the model of short rate of Vasicek. All it in order to equip the analyst with a right value that contemplates so much the own macroeconomic effects of the random movements of the interest rate like the process of diffusion with own jumps of the unfolding of a company.

VAN, Cash flow, Interest rate

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Approach to the Problem of Dynamic Stochastic Optimization (PDSO)

The expected present value, F_t, at the time t, of a project by calculating the discounted cash flows can be summarized as:

$$F_t = E\left[\int_t^\infty \phi_s e^{-\lambda u} du \mid \Im_t\right], \qquad (1.1)$$

Where ϕ_s represents the expected cash flow during the project development at time s, λ is the appropriate discount rate by sector and leverage ratio of the project and $\Im_t$ is all the relevant information (Avaliable) at the time t.

In this research, we propose to add to the value of the project, F_t, the value of the premium for "risk of ownership," which is to be modeled as a real option. To justify this approach, the existence of a risk adverse, infinitely lived investor who has access to a credit risk-free bond is assumed, B_t, whose performance (percentage change) is given by:

$$\frac{dB_t}{B_t} = r\, dt, \qquad (1.2)$$

Where r represents the risk free rate (default) paid for the bond. This agent also has access to a risky asset, i.e., the project, F_t, whose performance is given by a process of diffusion of the form:

$$dF_t = \left(\mu_F dt + \sigma_F dW_{1t}\right)F_t, \qquad (1.3)$$

Where μ_F is the average expected return of the project, σ_F is the project's instantaneous volatility and W_{1t} is a Brownian motion, i.e. $W_{1t} \sim N(0,t),$ in which case it holds that $E[dW_{1t}] = 0$ y $Var[dW_{1t}] = dt.$

In the case of expropriation, the almost unpredictable nature of the time of occurrence of the act of authority requires the modeling through American options, making it possible to assume that the investor's portfolio is comprised of long positions in a bond, B_t, that pays a risk-free rate and the risky asset, F_t, plus a short position in a call option on such asset, φ_t, this is:

$$\Pi_t = a_t(\omega_1 F_t + \omega_2 \varphi_t + \omega_3 B_t), \qquad (1.4)$$

Where ω_i represents the proportion of wealth that the investor assigns to each asset in its portfolio. The need to simplify the problem, leads to the assumption of a single date on which the government must decide whether or not to exercise the purchase option it has over the risky asset, making the American call on an European, which is analytically most treatable. Moreover, the non-financial nature of the underlying renders necessary the application of the methodology of real options for valuation, for further reference see: Abel (1983), Dixit and Pindyck (2000) or Trigeorgis (1996), to make a reference to the best known ones.

(1.3)

Broadly speaking, the real options methodology is based on applying the technology of financial options on contingent investment projects whose realization depends on the performance of a major project that serves as the underlying. Under this approach, the position and type of option is determined by the nature of the project analyzed, in the particular case of the risk of ownership, as explained above, it is a short position in a call option that is to say an actual closure option. The contingent nature of the option, ie, its derivative nature, leads to the use of Itô calculus in determining the stochastic differential equation that governs its premium. Applying the rules of stochastic calculus [1] to the performance of European call option, we get:

$$\frac{d\varphi}{\varphi_t} = \left(\mu_\varphi dt + \sigma_\varphi dW_{1t},\right),$$

(1.5)

That is nothing but the stochastic differential equation (SDE) of the performance of the closure's real option that models the "risk of ownership." In this case it holds that:

$$\mu_\varphi = \left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial F_t}\mu_F F_t + \frac{1}{2}\frac{\partial^2 \varphi}{\partial F_t^2}\sigma_F^2 F_t^2\right)\frac{1}{\varphi_t}$$

y

$$\left(\frac{\partial \varphi}{\partial F_t}\sigma_F\right)\frac{1}{\varphi_t} = \sigma_\varphi.$$

To complete the statement of the problem of dynamic stochastic optimization (PDSO), it is necessary to establish a profit function that reflects the increasing preference at decreasing rates for the benefits the agent analyzed presents.

For this, a profit function of the following form is proposed:

$$\Pi_t\left(B, F_t, \varphi_t\right) = \Pi_t^\gamma / \gamma \equiv \phi_t,$$

Which is supposed concave in accordance with risk averse agents.

PDSO Solution, structure of flat periods and returns on the underlying Brownian

After obtaining the differential equations that model the returns of the three assets that the investor has access to, we are in position to model the stochastic dynamic of the performance of their wealth, a_t, that is given by the following stochastic differential equation (SDE):

$$da_t = a_t\omega_1 dF_t + a_t\omega_2 d\varphi_t + a_t\left(1 - \left(\omega_1 + \omega_2\right)\right)dB_t - \Pi_t dt.$$

(1.6)

To determine the solution of the problem of maximizing (1.1) subject to (1.6), it is necessary to establish the optimal holdings of each of the assets to which the agent has access to, as well as the optimal expected profit[2]. For this you have to pose the maximization of the expected value of the discounted investor profits [3] by the appropriate rate, λ, subject to the budget constraint of their wealth, ie:

$$\text{Maximizar}_{\Pi_t} \quad \text{E}\left[\int_t^\infty \frac{\Pi_t^\gamma}{\gamma}e^{-\lambda u}du \mid \Im_t\right]$$

s. a.

$$da_t = a_t\omega_1 dF_t + a_t\omega_2 d\varphi_t + a_t\left(1 - \left(\omega_1 + \omega_2\right)\right)dB_t - \Pi_t.$$

(1.7)

Where $\Im_t$ represents all the relevant information available at the time *t*. To solve this problem (see, for example, Chiang (1992)), we resort to the value function.

$$J\left(a_t, t\right) = \max_{\Pi_t} \text{E}\left(\int_t^\infty \frac{\Pi_t^\gamma}{\gamma}e^{-\lambda s}ds \mid \Im_t\right).$$

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[1] As references see Lamberton and Lapeyre (1996), Mikosch (1998) and Gikhman y Skorokhod (2004).

[2] It must be remembered that profit is a random variable given the stochastic nature of asset returns.

[3] Typically proposed as the discount rate is the WACC (Weighted Average Capital Cost) of all assets from which the analyzed investor gets benefits.

From which the following recursive equation is obtained:

$$J\left(a_t,t\right)=\max_{\Pi_t}\mathrm{E}\left(\int_t^{t+dt}\frac{\Pi_t^\gamma}{\gamma}e^{-\lambda s}\mathrm{ds}+\int_{t+dt}^\infty\frac{\Pi_t^\gamma}{\gamma}e^{-\lambda s}\mathrm{ds}\,\middle|\,\mathfrak{I}_t\right).$$

(1.8)

After observing that the second addend within the expectation is the same functional J evaluated an instant after the starting point, if its value is approximated using the Fréchet differential and the mean value theorem of the integral in the first addend, we obtain:

$$0=\max_{\Pi_t,\omega_{1t},\omega_{2t}}\mathrm{E}\left[\left(\frac{\Pi_t^\gamma}{\gamma}e^{-\lambda s}+J_t+J_a a_t\left(r+\left(\mu_F-r\right)\omega_{1t}+\left(\mu_\varphi-r\right)\omega_{2t}-\frac{\Pi_t}{a_t}\right)+\frac{J_{aa}a_t^2}{2}\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)^2\right)\mathrm{d}t\right.$$
$$\left.+o\left(\mathrm{d}t\right)+J_a a_t\left(\sigma_F\omega_1+\sigma_\varphi\omega_2\right)\mathrm{d}W_{1t}\right].$$

(1.9)

If expectations are taken, it is divided over $\mathrm{d}t$ and the limit is taken when $\mathrm{d}t$ tend to zero, we obtain that:

$$0=\max_{\Pi_t,\omega_{1t},\omega_{2t}}\left[\frac{\Pi_t^\gamma}{\gamma}e^{-\lambda s}+J_t+J_a a_t\left(r+\left(\mu_F-r\right)\omega_{1t}+\left(\mu_\varphi-r\right)\omega_{2t}-\frac{\Pi_t}{a_t}\right)+\frac{J_{aa}a_t^2}{2}\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)^2\right].$$

(1.10)

Now, the separable function is proposed as the solution candidate $J\left(a_t,t\right)=\beta(a_t^\gamma/\gamma)e^{-\lambda t}$.

After performing some substitutions, a Hamiltonian of the following form is obtained:

$$0=\max_{\Pi_t,\omega_{1t},\omega_{2t}}\left[\frac{\Pi_t^\gamma}{\gamma}+\lambda\beta\frac{a_t^\gamma}{\gamma}+\beta a_t^\gamma\left(r+\left(\mu_F-r\right)\omega_{1t}+\left(\mu_\varphi-r\right)\omega_{2t}-\frac{\Pi_t}{a_t}\right)+\frac{\beta a_t^\gamma\left(\gamma-1\right)}{2}\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)^2\right].$$

(1.11)

Differentiating this expression with regard to each of the decision variables, $\Pi_t,\ \omega_{1t}\ y\ \omega_{2t},$ the following system of equations for the first order conditions (FOD) is obtained:

$$\frac{\partial H}{\partial\Pi_t}=0:\qquad\Pi_t^{\gamma-1}-\beta a_t^{\gamma-1}=0,$$

$$\frac{\partial H}{\partial\omega_{1t}}=0:\quad\beta a_t^\gamma\left(\mu_F-r\right)+\beta a_t^\gamma\left(\gamma-1\right)\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)\sigma_F=0,$$
1.8)

$$\frac{\partial H}{\partial\omega_{2t}}=0:\quad\beta a_t^\gamma\left(\mu_\varphi-r\right)+\beta a_t^\gamma\left(\gamma-1\right)\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)\sigma_\varphi=0.$$

(1.12)

If you use the last two equations, it is possible to determine that the premiums to the risk of the risky assets in the portfolio are identical, ie:

$$\frac{\mu_F-r}{\sigma_F}=\frac{\mu_\varphi-r}{\sigma_\varphi}.$$

(1.9)

If the mean and variance of the derivative (real option), given in equations (1.13) and (1.15) are replaced, the partial differential equation of second order of Black-Scholes (1973) is obtained, namely[4]:

$$0=\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial F}F_t r+\frac{1}{2}\frac{\partial^2\varphi}{\partial F^2}F_t^2\sigma_F^2-r\varphi.$$
1.10)

(1.13)

The above equation models the risk of ownership. Under the above assumptions, the boundary conditions of the real option are: $\max\left[F_T-K,0\right]$ and $\varphi_T\left(0,T\right)=0.$ The solution to (1.13) is given by:

$$\varphi_t=F_t N\left(d_1\right)-Ke^{-r\left(T-t\right)}N\left(d_2\right)$$
(1.11)

$$d_1=\frac{\ln\left(\frac{F_t}{K}\right)+\left(r+\frac{\sigma_F^2}{2}\right)\left(T-t\right)}{\sigma_F^2\sqrt{T-t}}\qquad d_2=d_1-\sqrt{T-t},$$

(1.14)

[4] See Venegas-Martínez (2008) or Neftci (2000).

Where the value of the "risk of ownership," φ_t, is expressed as a function of the value of the original project, F_t, the value of the compensation, K, the risk-free rate of default, r, the volatility of the original project, σ_F^2, and the time that the government has to make the decision, $T-t$. Once established, under the above assumptions, the solution of Black-Scholes as a proxy for the value of the "property risk" faced by the shareholders of a company, which represents a possible expropriation, the sensitivity of this risk facing changes in their incident factors can be established. Perhaps the most important of these factors is the compensation, K, to be paid by the government to those affected and can be established unilaterally by itself depending on his power or urgency. Making an anology with the financial options, it is viable to display the "*kappa*", $\kappa = \partial\varphi(\cdot)/\partial K$, as a measure of the change in the value of the premium received to a change in compensation that the government must provide to shareholders.

It can be demonstrated that $\kappa < 0$ for long positions in the options, so it is logical that for a short position it is taken that $\kappa > 0$, since this increases the probability of retaining the premium. To interpret this result, the analysis refers to the equation (1.4), which is accompanied by a positive sign to the ratio of wealth intended for the real option, which means that as the compensation increases, it is more likely for the shareholders to retain the value of the premium when the "nationalization" less attractive. Therefore, it is plausible to assume that the observed drop in market prices facing the announcement of an expropriation will be lower if a higher value of fixed compensation is set. Another factor is the volatility of the underlying, ie, of the original project which is represented by $\upsilon = \partial\varphi(\cdot)/\partial\sigma_F$.

Again, we can show a positive relationship between the volatility of the original project, σ_F, and the value of the option in a long position, $\varphi(\cdot)$, this is, Reversing the sign given the short position in the portfolio reviewed, we have: $\upsilon < 0$, which implies that the higher the volatility in the business the higher the increase in government incentives to control the company, thereby increasing the probability of exercising the real option, resulting in a negative effect on the value of the shares on the market; thereby undermining shareholder wealth. It is important to note that the solution to equation (1.13) is given by the familiar formula of Black-Scholes, which is valid if it has a structure of flat deadlines, complete markets, perfect divisibility of the underlying and normal distribution of returns of the underlying asset. Throughout this work the first and last cases will be flexibilized.

PDSO solution with Vasicek's cut rate and Brownian Yields of the Underlying

The next step in the extension of the modeling of the value of the "risk of ownership" is to suppose a flat term structure, which de facto represents a liquidity premium and produces more realism. For this a similar short rate to the one specified by Vasicek (1977) is used. The proposed model assumes that diffusion processes, both the interest rate, dW_{1t}, as well as the risky assets, dW_{2t}, are different but correlated. This assumption is a generalization of the developments in Mossin (1966), Sharpe (1964) and Treynor (1962), in which a relationship between free rate risk (with a structure of flat term) and performance of the risky assets is stablished, given by the known model CAPM. The relationships described above are represented by assuming that the short rate is driven by the EDE Vasicek, namely:

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$$\frac{dB_t}{B_t} = dr_t = \alpha(b - r_t)dt + \sigma_B dW_{2t},$$

(1.15)

where α represents the speed of adjustment of the short rate, r_t, to the long term rate, b, and σ_b is the volatility of the short rate. On the other hand, the EDE that governs risky asset performance follows the equation (1.13). As a consequence of his, EDE that leads the perfomance of the real option that models the "risk of ownership," is given by equation (1.15). The stochastic dynamic optimization problem the agent has to solve is similar to the one proposed in equation (1.17), except for the budget constraint, which leads to:

$$0 = \max_{\Pi_t, \omega_{1t}, \omega_{2t}} E \left[\begin{array}{c} \left(\frac{\Pi_t^\gamma}{\gamma} e^{-\lambda s} + J_t + J_a a_t \left(\alpha(b - r_t) + \left(\mu_F - \alpha(b - r_t) \right)\omega_{1t} + \left(\mu_\varphi - \alpha(b - r_t) \right)\omega_{2t} - \frac{\Pi_t}{a_t} \right) + \\ \frac{J_{aa} a_t^2}{2} \left(\sigma_B^2 (1 - \omega_1 - \omega_2)^2 + (\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t})^2 + 2\rho\sigma_B (1 - \omega_1 - \omega_2)(\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t}) \right) \right) dt \\ + o(dt) + J_a a_t \left((\sigma_F \omega_1 + \sigma_\varphi \omega_2)dW_{1t} + \sigma_B (1 - \omega_1 - \omega_2)dW_{2t} \right) \end{array} \right].$$

(1.16)

As before, you should take the expectation of equation (1.16), divide it by and taking the limit when it tends to zero, which gives:

$$0 = \max_{\Pi_t, \omega_{1t}, \omega_{2t}} \left[\begin{array}{c} \frac{\Pi_t^\gamma}{\gamma} e^{-\lambda s} + J_t + J_a a_t \left(\alpha(b - r_t) + \left(\mu_F - \alpha(b - r_t) \right)\omega_{1t} + \left(\mu_\varphi - \alpha(b - r_t) \right)\omega_{2t} - \frac{\Pi_t}{a_t} \right) + \\ \frac{J_{aa} a_t^2}{2} \left(\sigma_B^2 (1 - \omega_1 - \omega_2)^2 + (\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t})^2 + 2\rho\sigma_B (1 - \omega_1 - \omega_2)(\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t}) \right) \end{array} \right].$$

(1.17)

Again, you must propose a candidate solution of the form $J(a_t, t) = \beta(a_t^\gamma / \gamma)e^{-\lambda t}$ to determine the partial differential equation of second order governing φ. Then its partial derivatives are obtained and substituted into (1.17) to obtain the Hamiltonian, namely:

$$0 = \frac{\Pi_t^\gamma}{\gamma} + \frac{\lambda\beta a_t^\gamma}{\gamma} + \beta a_t^\gamma \left(\alpha(b - r_t) + \left(\mu_F - \alpha(b - r_t) \right)\omega_{1t} + \left(\mu_\varphi - \alpha(b - r_t) \right)\omega_{2t} - \frac{\Pi_t}{a_t} \right) +$$
1.15
$$\frac{(\gamma - 1)\beta a_t^\gamma}{2} \left(\sigma_B^2 (1 - \omega_1 - \omega_2)^2 + (\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t})^2 + 2\rho\sigma_B (1 - \omega_1 - \omega_2)(\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t}) \right).$$

(1.18)

From this Hamiltonian, it is possible to obtain the first order conditions (Necessary conditions) deriving it with respect to the variables whose optimum is sought, namely:

$$\frac{\partial H}{\partial \Pi_t} = 0: \qquad \Pi_t^{\gamma-1} - \beta a_t^{\gamma-1} = 0$$

$$\frac{\partial H}{\partial \omega_{1t}} = 0: \qquad \begin{array}{l} \beta a_t^\gamma \left(\mu_F - \alpha(b - r_t) \right) + \frac{\beta a_t^\gamma (\gamma - 1)}{2} \left(-2\sigma_B^2 (1 - \omega_1 - \omega_2) + 2(\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t})\sigma_F \right) + \\ \frac{\beta a_t^\gamma (\gamma - 1)}{2} \left(2\rho\sigma_B \left((1 - \omega_1 - \omega_2)\sigma_F - (\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t}) \right) \right) = 0 \end{array}$$

$$\frac{\partial H}{\partial \omega_{2t}} = 0: \qquad \begin{array}{l} \beta a_t^\gamma \left(\mu_\varphi - \alpha(b - r_t) \right) + \frac{\beta a_t^\gamma (\gamma - 1)}{2} \left(-2\sigma_B^2 (1 - \omega_1 - \omega_2) + 2(\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t})\sigma_\varphi \right) + \\ 1.16 \\ \frac{\beta a_t^\gamma (\gamma - 1)}{2} \left(2\rho\sigma_B \left((1 - \omega_1 - \omega_2)\sigma_\varphi - (\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t}) \right) \right) = 0. \end{array}$$

(1.19)

The first important result of this extension is weak corroboration of Fisher's separation theorem; see in this respect, Fisher (1930). What in fact means that the policy of dividend payments, as a proportion of wealth is independent of the investment policy of the firm provided that the deposit rates are equal to the active for equal installments and unrestricted investment opportunities. It is said that corroboration is weak, since the value of β in terms of the proportions of wealth allocated to each asset, which change before changes in the budget constraint, remain constant. This statement corresponds to the equality between the first equations of the CPOs of the two PDSOs despite changes in the risk-free rate, the inclusion of a new source of uncertainty in the problem and the consequent alteration of the course of the wealth of the individual.

The second important result of this exercise, is given by equalizing the past two CPOs so that the premiums to the risk of both assets are equal, that is:

$$\frac{\left(\mu_F-\alpha(b-r_t)\right)-(1-\gamma)\sigma_B^2\left(1-\omega_1-\omega_2\right)-\rho\sigma_B\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)}{\sigma_F}=(1-\gamma)\left(\sigma_F\omega_1+\sigma_\varphi\omega_2+\rho\sigma_B\left(1-\omega_1-\omega_2\right)\right)$$

$$\frac{\left(\mu_\varphi-\alpha(b-r_t)\right)-(1-\gamma)\sigma_B^2\left(1-\omega_1-\omega_2\right)-\rho\sigma_B\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)}{\sigma_\varphi}=(1-\gamma)\left(\sigma_F\omega_1+\sigma_\varphi\omega_2+\rho\sigma_B\left(1-\omega_1-\omega_2\right)\right),$$

(1.20)

Which in turn can be rewritten as:

$$\frac{\mu_F-\alpha(b-r_t)-\eta}{\sigma_F}=\frac{\mu_\varphi-\alpha(b-r_t)-\eta}{\sigma_\varphi},$$

Where:

$$\eta=(1-\gamma)\sigma_B^2\left(1-\omega_1-\omega_2\right)+\rho\sigma_B\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right).$$

The above equality can be brought to a partial differential equation of second order.

In effect, after replacing μ_φ and σ_φ in the above equation, we have:

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\frac{\partial^2\varphi}{\partial F_t^2}F_t^2\sigma_F^2+\frac{\partial\varphi}{\partial F_t}F_t\alpha(b-r_t)-\varphi\alpha(b-r_t)+\frac{\partial\varphi}{\partial F_t}F_t\eta-\eta\varphi=0,$$

(1.21)

Where $\alpha(b-r_t)$ is the deterministic part (trend) of the short rate and $(\partial\varphi/\partial F_t)F_t\eta-\eta\varphi$ is added in response to the new source of uncertainty provided by the short rate specified in equation (1.15).

This result implies that the real option modeling the "risk of property" is beyond the Black-Scholes formula, because of the last two terms. For its solution, you can always resort to numerical methods or the Monte Carlo simulation method.

If you want to know the optimal proportions of wealth allocated to each asset, it is necessary to start from the system of equations expressed in (1.20) and denote:

$$\lambda_i=\frac{\mu_i-\alpha(b-r_t)}{\sigma_i}\quad(1.20)$$

As the Premium to the risk (market) paid to each risky asset, from which it is obtained by matching that:

$$\lambda_F-\frac{1-\gamma}{\sigma_F}\left(\sigma_B^2-\sigma_B^2\omega_1-\sigma_B^2\omega_2-\sigma_F\omega_1-\sigma_F\omega_2\right)=\lambda_\varphi-\frac{1-\gamma}{\sigma_\varphi}\left(\sigma_B^2-\sigma_B^2\omega_1-\sigma_B^2\omega_2-\sigma_F\omega_1-\sigma_F\omega_2\right),$$

(1.22)

From where we obtain:

$$\omega_1=\frac{\left(\lambda_F-\lambda_\varphi\right)\sigma_F\sigma_\varphi}{\left(\sigma_B^2+\sigma_F\right)(1-\gamma)\left(\sigma_\varphi-\sigma_F\right)}-\omega_2\left(\sigma_B^2+\sigma_\varphi\right)-\sigma_B^2.$$

If this expression is substituted in and if we denote.

$$\Omega=\frac{\left(\lambda_F-\lambda_\varphi\right)\sigma_F\sigma_\varphi}{\left(\sigma_B^2+\sigma_F\right)(1-\gamma)\left(\sigma_\varphi-\sigma_F\right)},\quad(1.21)$$

We obtain that:

$$\omega_2=\frac{\dfrac{\lambda_F}{1-\gamma}+\Omega\left(\rho\sigma_B-\dfrac{\sigma_B^2}{\sigma_F}\right)+\sigma_B^2\left(\dfrac{\sigma_B^2+\sigma_F+1}{\sigma_F}-\sigma_F-\rho\sigma_B\right)-\rho\sigma_B}{-\left(\left(\sigma_B^2+\sigma_F+1\right)\left(\dfrac{\sigma_B^2}{\sigma_F}-\rho\sigma_B\right)+\left(\sigma_B^2+\sigma_\varphi\right)\left(-\sigma_F-1\right)+\sigma_\varphi\left(1-\dfrac{1}{\sigma_F}\right)\right)}.$$

The value of ω_1 can be obtained substituting ω_2 in any of the equations of the system given in (1.20).

ECONOMY

Solution of the PDSO with Vasicek's short rate and performance correlated with jumps

For the last stage of the theoretical analysis of the work, we will suppose an environment of financial crisis in which the performance of the company subject to expropriation, may undergo abrupt jumps that are outside the explanatory power of the purely Brownian diffusion proceses, which is outside of the scope of the traditional methodology of real options. The presence of these jumps will be modeled by a Poisson distribution with intensity λ and average jump size equal to ς, which will allow the existence of "extreme" performances that are beyond what was forecast by a normal distribution, ie.

Beyond the assumption of a Brownian diffusion process. Although it is known that the quarterly yields of low frequency, Vg., are distributed as normal random variables, it is interesting to note that under conditions of financial crisis, yields have jumps that can not be explained by the normal distribution. Given the empirical evidence, we have simulated a group of performances governed by a diffusion process with jumps whose jump threshold is given by the probability of a random variable that follows a Poisson distribution that can only present a jump per period of timeIn this case, the average jump size is obtained as the average excess of the yields, ie yields above two standard deviations from a normal distribution, which coincides with what is traditionally considered as an excess after the VaR; this is an abnormal performance. For this analysis, it is valid to start with the PDSO's approach when the underlying follows a jump-diffusion process. The approach to this problem is similar to the above, the only change to be made is given by the stochastic differential equation followed by the yields of the risky asset in a crisis, namely:

$$dF_t = \left(\mu_F dt + \sigma_F dW_{1t} + \zeta dN_t \right) F_t. \tag{1.23}$$

Where a Poisson jump process was added to the diffusion process, (3) dN_t, which on average jumps once at the instant dt with a probability (proportional to the average intensity λ.)

So the probability of a jump not happening is given by $1 - \lambda dt + o(dt)$, and the probability of more than one jump per unit of time is given by $o(dt)$.

The inclusion of this new element responds to the need to model the abrupt change in the yields of the company caused by the rumor of government intervention. In general, it is untimely and the effects can be, given the quarterly basis, encompassed in a single event, which coincides fully with the modeling through Poisson jumps. In this case, it is possible to demonstrate that this infinitesimal of higher order than the first tends to zero as the study interval collapses to the same point, that is:

$$\lim_{dt \to 0} \left[\frac{o(dt)}{dt} \right] = 0.$$

Similarly, it can be demonstrated that:

$$E[dN_t] = Var[dN_t] = \lambda dt.$$

Where μ_φ and σ_φ are given like in (1.5), while the stochastic differential equation governing the short rate, is still expressed in (1.15), ie, the model of Vasicek (1977).

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The above equation reflects the inclusion of the Poisson process (jump) in the differential equation that governs the yields of the real option through which the expropriation risk is modeled.

In general, this option has no value (is deeply out of the money), except for the moments in which a credible rumor about nationalization "activates" the jump component, dN_t, and atkes it to levels where it existence affects the value of the implicit portfolio [5] of the shareholders.

Again, in the problem statement the changes made on the equations that govern the yields of the risky asset and the derivative will be noticeable until the differential functional is obtained, $dJ(a_t,t)$. The reason for this change is the inclusion of the restriction (which includes the "active" option modeling the risk of ownership) in the search for the optimum. To solve the problem at hand, we have:

$$0 = \max_{\Pi_t, \omega_{1t}, \omega_{2t}} E \left[\begin{array}{c} \left(\frac{\Pi_t^\gamma}{\gamma} e^{-\lambda s} + J_t + J_a a_t \left(\alpha(b-r_t) + (\mu_F - \alpha(b-r_t))\omega_{1t} + (\mu_\varphi - \alpha(b-r_t))\omega_{2t} - \frac{\Pi_t}{a_t} \right) + \right. \\ \left. \frac{J_{aa} a_t^2}{2} \left(\sigma_B^2 (1-\omega_1-\omega_2)^2 + (\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t})^2 + 2\rho\sigma_B (1-\omega_1-\omega_2)(\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t}) \right) \right) dt + o(dt) \\ + J_a a_t \left((\sigma_F \omega_1 + \sigma_\varphi \omega_2) dW_{1t} + \sigma_B (1-\omega_1-\omega_2) dW_{2t} \right) + J_a a_t \left(v\omega_{1t} + (\varphi(F_t(1+v),t) - \varphi(F_t,t))\omega_{2t} \right) dN_t \end{array} \right].$$

(1.25)

Again, it is neccesary to take the expectation of the above expression and taking its limit when the analyzed interval collapses to toz ero, which gives:

$$0 = \max_{\Pi_t, \omega_{1t}, \omega_{2t}} \frac{\Pi_t^\gamma}{\gamma} e^{-\lambda s} + J_t + J_a a_t \left(\alpha(b-r_t) + (\mu_F - \alpha(b-r_t))\omega_{1t} + (\mu_\varphi - \alpha(b-r_t))\omega_{2t} + \zeta(\omega_{2t} + \omega_{1t}) - \frac{\Pi_t}{a_t} \right) + \frac{J_{aa} a_t^2}{2} \left(\sigma_B^2(1-\omega_1-\omega_2)^2 + (\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t})^2 + 2\rho\sigma_B(1-\omega_1-\omega_2)(\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t}) \right),$$

(1.26)

Note that this expression differs from the previous similar PDSO only in the average value of the jump, $\zeta(\omega_{2t} + \omega_{1t})$, which is added to the expectation of the partial derivative with respect to the wealth of the functional, $J_a(a_t,t)$.

Indeed, the average jump size in the value of assets under expropriation risk affects the average investor's wealth in $J_a a_t \zeta(\omega_{2t} + \omega_{1t})$ units, not its variance, thereby modifying its budget constraint and, therefore, its affordable set of benefits.

By proceeding with the solution of the PDSO it is necessary to establish as a candidate of solution, $J(a_t,t) = \beta \frac{a_t^\gamma}{\gamma} e^{-\lambda t},$ for (1.26). Since the objective function remains unchanged in all three approaches, you can use the same candidate to a solution (1.25) for them all, so the following Hamiltonian is obtained, where only the average wealth was altered as follows:

$$0 = \frac{\Pi_t^\gamma}{\gamma} + \frac{\lambda\beta a_t^\gamma}{\gamma} + \beta a_t^\gamma \left(\alpha(b-r_t) + (\mu_F - \alpha(b-r_t))\omega_{1t} + (\mu_\varphi - \alpha(b-r_t))\omega_{2t} + \zeta(\omega_{1t} + \omega_{2t}) - \frac{\Pi_t}{a_t} \right) + \frac{(\gamma-1)\beta a_t^\gamma}{2} \left(\sigma_B^2(1-\omega_1-\omega_2)^2 + (\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t})^2 + 2\rho\sigma_B(1-\omega_1-\omega_2)(\sigma_F \omega_{1t} + \sigma_\varphi \omega_{2t}) \right).$$

(1.27)

After taking the partial derivatives of the Hamiltonian with respect to each of the control variables, the following set of conditions is achieved:

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[5] In the literature of asset valuation with real options it is established that any project has a number of embedded real options, Vg. expansion, closure, postponement, etc.., affecting the value of the project. Overall, the project can be seen as a portfolio of risky assets (the project) and a number of options on it.

$$\frac{\partial H}{\partial \Pi_t}=0: \qquad \Pi_t^{\gamma-1}-\beta a_t^{\gamma-1}=0$$

$$\frac{\partial H}{\partial \omega_{1t}}=0: \quad \begin{aligned}&\beta a_t^\gamma\left(\mu_F-\alpha(b-r_t)+\zeta\right)+\frac{\beta a_t^\gamma(\gamma-1)}{2}\left(-2\sigma_B^2(1-\omega_1-\omega_2)+2\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)\sigma_F\right)+\\ &\frac{\beta a_t^\gamma(\gamma-1)}{2}\left(2\rho\sigma_B\left((1-\omega_1-\omega_2)\sigma_F-\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)\right)\right)=0\end{aligned}$$

$$\frac{\partial H}{\partial \omega_{2t}}=0: \quad \begin{aligned}&\beta a_t^\gamma\left(\mu_F-\alpha(b-r_t)+\zeta\right)+\frac{\beta a_t^\gamma(\gamma-1)}{2}\left(-2\sigma_B^2(1-\omega_1-\omega_2)+2\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)\sigma_\varphi\right)+\\ &\frac{\beta a_t^\gamma(\gamma-1)}{2}\left(2\rho\sigma_B\left((1-\omega_1-\omega_2)\sigma_\varphi-\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)\right)\right)=0.\end{aligned}$$

$$(1.28)$$

As in the two approaches of the previous sections, we can see that the optimal path of expected dividends. Given by the first partial derivative remains unchanged, namely:
$$\Pi_t=\beta^{1/(\gamma-1)}a_t,$$
which confirms again the weak enforcement of the Fisher theorem. Note that the inclusion of a Poisson jump correlated with asset returns affects only the best proportions of options and underlying, ω_1 y ω_2, that the investor should keep in their portfolio, not their benefits.
This solution is in that, in a time of uncertainty caused by the announcement of a possible expropriation, the investor changes the delta coverage ratio of the risky asset. The amount and direction of this change will be the result of the market perception of the expropriation, ie, on the value of the jump, ς. Now it only remains to find the partial differential equation governing the second order derivative price necessary to cover the risk for property under the assumptions explained in the PDSO resolved above. In fact, matching the Hamiltonian partials regarding the optimal proportions, ω_{1t} and ω_{2t}, we obtain that:

$$\frac{\left(\mu_F-\alpha(b-r_t)\right)+\zeta-(1-\gamma)\sigma_B^2(1-\omega_1-\omega_2)-\rho\sigma_B\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)}{\sigma_F}=(1-\gamma)\left(\sigma_F\omega_1+\sigma_\varphi\omega_2+\rho\sigma_B(1-\omega_1-\omega_2)\right)$$

$$\frac{\left(\mu_\varphi-\alpha(b-r_t)\right)+\zeta-(1-\gamma)\sigma_B^2(1-\omega_1-\omega_2)-\rho\sigma_B\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right)}{\sigma_\varphi}=(1-\gamma)\left(\sigma_F\omega_1+\sigma_\varphi\omega_2+\rho\sigma_B(1-\omega_1-\omega_2)\right).$$

$$(1.29)$$

This implies that the risk premium of the real option and the underlying asset are, as in the previous exercises, the same. In this case, the risk premium is affected by both the volatility of the assets in the portfolio and by the short rate correlation with the risk of expropriation associated with the underlying, ρ.

After matching the above equations, and substituting μ_φ and σ_φ, a partial differential equation of the second order similar to that obtained in previous PDSO's is obtained, that is:

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\frac{\partial^2\varphi}{\partial F_t^2}F_t^2\sigma_F^2+\frac{\partial\varphi}{\partial F_t}F_t\alpha(b-r_t)-\varphi\alpha(b-r_t)+\frac{\partial\varphi}{\partial F_t}F_t\eta_\zeta-\eta_\zeta\varphi,$$

$$(1.30)$$

Where:

$$\eta_\zeta=\zeta+(1-\gamma)\sigma_B^2\left(1-\omega_1-\omega_2\right)+\rho\sigma_B\left(\sigma_F\omega_{1t}+\sigma_\varphi\omega_{2t}\right).$$

As can be seen in the above expression, the new PDE (Partial Differential Equation) incorporates the average value of the jump, ς, maintaining the traditional form of the PDE's followed by derivatives. It is also necessary to note that the PDE from previous optimization exercise is nonlinear and its solution is beyond the scope of the proposal by Black and Scholes. Similarly, its solution is beyond the scope of the methods traditionally used in the valuation of real options, since the presence of jumps precludes the use of recombinant trees. It is important to note that so far we have only talked about the average value of the jump size, ς, without mentioning the distribution of this one. This issue has been put aside intentionally as there are specific cases that shed closed solutions, eg. if the jump size is distributed as a log-normal random variable, Merton (1976) finds that the solution to the parabolic partial differential equation is given by:

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$$C_{jd} = \sum_{n=0}^{\infty} \frac{e^{\lambda(1+\kappa)(T-t)}\left(\lambda(1+\kappa)(T-t)^n\right)}{n!} C_{BS}\left(S,X,r_n,\sigma_n^2,T-t\right)$$

(1.32)

Where $\lambda,$ is the parameter of the intensity of the jumps process $C_{BS}(\cdot),$ e is the value of a Black-Scholes purchase option, it is the average of the size distribution of the jump, X is the exercise price,

$\sigma_n^2 = \sigma^2 + \dfrac{n\delta^2}{T-t},$ is the volatility of the underlying y $r_n = r - \lambda\kappa + \dfrac{n\ln(1+\kappa)}{T-t}.$

This particular solution to the problem posed by (1.30) represents the value of the premium for the risk of expropriation. We recommend taking this approach with caution as the nature of the phenomenon under study implies the possibility of extreme values in the jump size.

The solution of the PDE, in this section, is the basis of the valuation of the assets of a banking institution in Mexico, it is assumed that the yields of the action (global) are correlated with the rate of risk free rate of the United States and its performance was affected by the notice of the intention of the U.S. government to become the largest shareholder of the company. The next step will be the reply to the previous theoretical exercise data from a Mexican company.

References

Barone-Adesi, G and R. E. Whaley (1987). "Efficient Analytic Approximation of American Option Values". *Journal of Finance*, Vol. 42, No. 2, pp 301-320.

Chiang, A, (2000), Elements of Dynamic Optimization, Waveland Press, NY, USA, p. 327.   (1.31)

Dixit, A. K. and R. S. Pindyck (1994). Investment under Uncertainty, Princeton University Press, Princeton, N. J.

Dixit, A. K. and R. S. Pindyck (1995). "The Options Approach to Capital Investment". *Harvard Business Review*, May-June.

Dixit, A. K. and R. S. Pindyck (2000). Expandability, Reversibility, and Optimal Capacity Choice, Project Flexibility, Agency, and Competition, New Developments in the Theory and Applications of Real Options, Eds. Brennan & Trigeorgis, Oxford University Press.

Faccio, M., R. Masulis, and J. J. McConnell (2006), "Political Connections and Corporate Bailouts", Journal of Finance, Vol. 61. No. 6. pp. 2597-2635.

Fisher, I (1930). "The Theory of Interest: As determined by impatience to spend income and opportunity to invest it". 1954 reprint, New York: Kelley and Millman.

Gikhman, I. I, Skorokhod, A and S. Kotz (2004). The Theory of Stochastic Processes I, Springer, Berlin, p. 574.

Girsanov, I.V. (1960). "On transforming a certain class of stochastic processes by absolutely continuous substitution of measures". *Journal of Theory of Probability and its Applications*, Vol. 5, pp 285-301.

Henderson, V., and D. G. Hobson (2002). "Real Options with Constant Relative Risk Aversion". *Journal of Economic Dynamics and Control*, Vol. 27. No. 2, pp. 329-355.

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Lamberton, D. and B. Lapeyre (1996). Introduction to Stochastic Calculus Applied to Finance, Chapman & Hall Eds., (English version of 1991 original French version).

Merton, R.C. (1976). "Options pricing when Underlying Stock Returns are Discontinuous". *Journal of Financial Economics*, Vol. 3, No. 2, pp. 125 -144.

Mikosch, T. (1998): Elementary Stochastic Calculus with Finance in View", World Scientific Publishing.

Mossin, J. (1966). "Equilibrium in a Capital Asset Market". *Econometrica*, Vol. 34, No. 4, pp. 768-783.

Neftci, S. N. (2000), "An introduction to the Mathematics of Financial Derivatives", Academic Press, NY, USA, 527 p.

Rosas, G. (2006). "Bagehot or Bailout? An Analysis of Government Responses to Banking Crises". *American Journal of Political Science*, Vol. 50, No. 1, pp. 175-191.

Sharpe, W. F. (1964). "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk". *Journal of Finance*, Vol. 19, No. 3. pp. 425-442.

Strobel, F. (2005). "Monetary Integration and Inflation Preferences: A Real Options Analysis. *European Economic Review*, Vol. 49, No. 4, pp. 845-860.

Schwartz, E. S. and L. Trigeorgis (2001), Real Options and Investment under Uncertainty. The MIT Press Cambridge, Massachusetts London, England.

Schneider, M. and A. Tornell (2004). "Balance Sheet Effects, Bailout Guarantees and Financial Crises". *The Review of Economic Studies*, Vol. 71, No. 3, pp. 883-913.

Treynor, J. L. (1962). Toward a Theory of Market Value of Risky Assets. Unpublished manuscript. A final version was published in 1999, in Asset Pricing and Portfolio Performance: Models, Strategy and Performance Metrics. Robert A. Korajczyk (editor) London: Risk Books.

Trigeorgis, L. (1996), Real Options: Managerial Flexibility and Strategy in Resource Allocation, MIT Press, 421 p.

Vasicek, O. (1977). "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, Vol. 5, No. 2, pp 177-188.

Venegas-Martínez, F. (2008): Riesgos financieros y economicos, productos derivados y decisiones economicas bajo incertidumbre, 2da. Edicion, Cengage Learning (anteriormente Thomson), México.

Verret, J. W. (2009). "Treasury Inc.: How the Bailout Reshapes Corporate Theory & Practice". http://ssrn.com/abstract=1461143.

Villalón, J (1991), "Matematicas de las decisiones financieras y sus aplicaciones", Tomo I, Ed. Centro de Estudios Ramon Areces, S. A., España.

Páginas web consultadas
http://www.cnbv.gob.mx/default.asp?com_id=2